On the Structure of $K_{2,r}$ Minor Free Graphs

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RÉSUMÉ. Résumé: Dans ce papier nous nous intéressons à l'étude de la structure des graphes qui excluant $K_{2,r}$ comme mineur. Nous montrons que tout graphe sans mineur $K_{2,4}$ admet deux sommets dont la suppression rend le graphe planaire-extérieur. Ces sommets peuvent extraites en temps linéaire. Si le graphe est planaire, un sommet suffit à le rendre planaire-extérieur. Une des conséquence de ce résultat est que les graphes sans mineur $K_{2,4}$ sont de largeur arborescente au plus 4, et au plus 3 pour le cas planaire. Ces bornes sont optimale à cause de K_5 et de K_4. Nous mettons aussi en évidence la relation entre la taille de la grille de la plus petite grilles permettant de dessiner un graphe H et le graphe excluant H comme mineur. Une conséquence de ce résultats est que la largeur arborescente des graphes sans mineur $K_{2,r}$ est $O(\sqrt{r})$ et $r^{1/2+o(1)}$ s'il exclue un graphe planair-exterieur à r sommets.

ABSTRACT. Abtract: This paper concerns the structure of the graphs excluding a $K_{2,r}$-minor. We prove that every $K_{2,4}$-minor free graph contains two vertices whose removal leaves the graph outerplanar. Such vertices can be founded in linear time. If the graph is planar the removal of one vertex suffices to leave the graph outerplanar. It follows that the treewidth of $K_{2,4}$-minor free graph is at most 4, and at most 3 for the planar case. These bounds are optimal because of K_5 and K_4. We also establish a connection between the size of a poly-line grid drawing of a given planar graph H and the treewidth of any planar H-minor free graph. A consequence is that the treewidth of planar $K_{2,r}$-minor free graphs is $O(\sqrt{r})$, and $r^{1/2+o(1)}$ if one excludes any r-vertex outerplanar graph.

MOTS-CLÉS : Mots clefs: graphe sans mineur, largeur-arborescente, graphes planaires

KEYWORDS : Keywords: minor free graphs, treewidth, planar graphs, outerplanar graphs

1. Introduction

Graph decomposition plays an important role in graph algorithmic. Tree-decompositions, introduced in [1] and rediscovered by[2], is a very popular one. They are central in the Fixed Parameterized Tractable (FPT) Theory [3] whose consequences for practical algorithms are effective improvements on the running time [4, 5, 6]. Arbitrary graphs have no fine structure in general, however graphs excluding some minors[1] have one.

One can say a lot about the structure if the excluded minor is small. A graph excluding a K_3-minor is a forest, and a graph excluding K_4-minor has treewidth at most two. Roughly speaking a tree-decomposition is a collection of subgraphs, called bags, that cover the graph in a tree-like manner (see [7] for precise definitions). Treewidth-k graphs are graphs having a tree-decomposition into bags of at most $k+1$ vertices. The structure of K_5-minor free graphs, given by Wagner [8], can be expressed in term of tree-decomposition into bags composed of either of planar graphs, or of V_8-graphs, a cubic graph on eight vertices. The treewidth of such graphs is unbounded in general.

Actually, Roberston and Seymour [9] showed that every graph excluding a fixed minor H has a tree-decomposition into bags almost-embeddable on a surface on which H cannot be embedded. The above results about graphs excluding K_3, K_4, or K_5 is therefore captured by this general result. However, the exact structure cannot be derived since extremely large constants, depending on H, are involved. In fact, no upper bounds on these constants is known.

Nevertheless, such general tree-decompositions have lead to important algorithmic applications. Among them we just point [10] for additively approximating the chromatic number of a graph, and [11] for shortest-path decomposition and its applications to distance oracles and routing schemes.

Determining the fine structure of minor free graphs becomes much more complicated when the excluding graph has more than five vertices. The structure of K_r-minor free graphs is still open for $r > 5$, and we refer to [12] for a further discussions. Among simple open problems about K_6-minor free graphs, let us mention the maximum arboricity (conjectured to be 3), and the Jørgensen's conjecture in relation with the Hadwiger's conjecture :

Conjecture 1 (Jørgensen, 1994) *Every 6-connected K_6-minor free graph G has a vertex u such that $G \setminus \{u\}$ is planar.*

1.1. Our Results

We show that every 2-connected $K_{2,4}$-minor free graph G has two vertices u, v such that $G \setminus \{u, v\}$ is outerplanar. More precisely,

Theorem 1 *In every 2-connected graph with n vertices, we can in $O(n)$ time either extract a $K_{2,4}$-minor, or find two vertices (one if the graph is planar) whose removal leaves the graph outerplanar.*

We may naturally wonder whether Theorem 1 can be extended to $K_{2,r}$ with $r > 4$. As we will see in Section 3, there are $K_{2,5}$-minor free graphs with n vertices, that are planar and 2-connected, such that the removal of at least $\Omega(n)$ vertices is required to make the graph outerplanar. [A RING OF K_4 SUFFICES ? ? ?] So the only way to extend this theorem to $r > 4$ is to assume higher connectivity.

An immediate corollary of Theorem 1 is that $K_{2,4}$-minor free graphs have treewidth at most 4, and at most 3 if the graph is planar, and the corresponding tree-decomposition can be constructed in linear time. Indeed, a $K_{2,4}$-minor of a graph G must wholly appear in a 2-connected component of G, and the treewidth of G is the maximum over the treewidth of its 2-connected components. These two bounds are

1. A minor of a graph G is a subgraph of a graph obtained from G by edge contraction.

best possible because of K_5, and of K_4 for planar graphs. It improves the treewidth upper bounds of 6 given by Bodlaender et al. [13], and Thilikos [14] for the planar case.

The treewidth of planar $K_{2,r}$-minor free graphs is later discussed in Section 4. We prove a bound of $O(\sqrt{r})$, an asymptotically improvement upon the $r+2$ upper bound of [14]. Actually, we establish a connection between the treewidth of a H-minor free graphs and the ability of poly-line grid[2] drawing of H. More precisely, we show :

Theorem 2 *The treewidth of every planar graph excluding a graph having a poly-line $p \times q$-grid drawing is $O(p^{3/2}\sqrt{q})$.*

Because $K_{2,r}$ can be drawn on a $3 \times r$ grid (see Fig. 4), it follows that the treewidth of planar $K_{2,r}$-minor free graphs is $O(\sqrt{r})$. This later bound is asymptotically optimal because of the $k \times k$ grid that has treewidth k and clearly excludes K_{2,k^2}. We derive similar bounds on the treewidth of planar graphs excluding a tree of a given pathwidth or an outerplanar graph (see Section 4).

1.2. Related Works

The study of graphs excluding a given graph as a minor has long history. The structure of graphs excluding a minor is known for K_5 and $K_{3,3}$ [8], the octahedron plus an edge and the 3-cube [15, 16] if the graph is enough connected. New characterizations of graphs excluding as a minor a K_5, or an octahedron has been given in [17]. The treewidth of graphs excluding as minor an r-vertex planar graph is $2^{O(r^5)}$ [18], and is conjectured to be $r^{2+o(1)}$. It reduces to $O(r)$ for excluding (as minor) a $K_{2,r}$ [13], a forest [19] or a cycle [20] on r vertices, $O(r^2)$ for excluding r disjoint triangles [21] or a r-prism[3] graph [22], and $O(r^3)$ for excluding simultaneously a 2-row grid and a circus graph[4] [23], see also [24]. More specifically, the treewidth of graphs excluding a 3×3-grid is at most 7 which is optimal because of K_8 [25].

There are also works that study graphs containing fixed minors. Among them, [26] have showed that, for every integers s, r, there is a number $N(s,r)$ such that every $\frac{31}{2}(k+1)$-connected graph with at least $N(s,r)$ vertices contains a $K_{s,r}$-minors. Recently, several authors have announced that any large 5-connected graph contains a $K_{2,k}$-minor, and similar conditions forcing $K_{3,r}$ and $K_{4,r}$ minors.

The maximum density of a $K_{2,r}$-minor free graph has been determined by Chudnovsky, Reed and Seymour [27]. More precisely, for every $r \geqslant 1$, the densest $K_{2,r}$-minor free graph with n vertices has $\frac{1}{2}(n-1)(r+1)$ edges. The highest density of $K_{s,r}$-minor free graphs is studied in [28], but only partial answers are known whenever $s > 2$.

Motivated by routing problems, Bodlaender et al. [13] have studied the treewidth of graphs excluding a $K_{2,r}$-minor. They show an upper bound of $2r-2$, that reduces to $r+2$ if the graph is planar [14].

The problem of determining whether H is a minor of G is computationally difficult for general H (e.g. if H is a cycle of $|V(G)|$ vertices). However the problem is FPT in H. There is a cubic time algorithm (for fixed H), and a linear time algorithm if $H = K_5$ of H is planar (cf. [29]). In particular this is linear for $H = K_{2,r}$ and fixed r.

2. The Structure of Graphs Excluding $K_{2,4}$

It is worth to say that the family of $K_{2,4}$-minor free graphs includes non-planar graphs, K_5 and $K_{3,3}$ are such examples. In order to prove Theorem 1 we combine the following two intermediate results that basically distinguish the planar and non-planar case.

2. A $p \times q$-grid is a mesh of p rows and q columns.
3. The product of a r-vertex cycle and a K_2. In particular, graphs with no 12-prism minor have treewidth at most 7262 and exclude a 4×4-grid, a huge improvement upon the 2^{20r^5} upper bound for $r \times r$-grid [18] which is $2^{20,480}$ for $r = 4$.
4. A minor of a 3-row grid obtained by contracting the first row and removing all edges of the second row.

Lemma 1 *If G is a planar 2-connected graph, then in linear time we can either extract from G a $K_{2,4}$-minor, or find a vertex whose removal leaves G outerplanar.*

Lemma 2 *If G is a non-planar 2-connected graph with n vertices, then in $O(n)$ time we can either extract a $K_{2,4}$-minor, or find a vertex whose removal leaves G planar.*

2.1. Preliminaries

Consider a plane graph G, that is an embedding of graph G in the the plane $\mathbb{R}^2$. The connected subsets of $\mathbb{R}^2 \setminus G$ are the *faces* of G, and each one, except the infinite one a.k.a. the outerface, is homeomorphic to an open disc. An embedding is *outerplane* if all the vertices lie on the border of the outerface.

For a path M of G, we denote by $M[u,v]$ the subpath of M going from vertex u to vertex v. We set $M[u,v[= M[u,v] \setminus \{v\}$, $M]u,v] = M[u,v] \setminus \{u\}$, and $M]u,v[= M[u,v] \setminus \{u,v\}$. The *length* $|M|$ of M is its number of edges. Notations extend to plane cycles as follows. For a cycle C of G, we denote by $C^+[u,v]$ the path going clockwise from u to v along C, and we define similarly the variants $C^+[u,v[$, $C^+]u,v]$, and $C^+]u,v[$.

For a subgraph H of G, denote by $\textsc{In}(H) \subset \mathbb{R}^2$ the subset $\mathbb{R}^2 \setminus H$ where the outerface is excluded. If H is a simple cycle, then $\textsc{In}(H)$ consists of one region (or face of H) whose border is H. By extension, $\textsc{In}(H)$ denotes also the subgraph of G induced by the vertices that belong to $\textsc{In}(H)$. An *attachment* of a connected component X of $G \setminus H$ is a vertex of H adjacent to a vertex of X.

Whenever we extract a minor K from G, we actually construct a *model* of K defined as follows : with each vertex u of K we associate a *super-node*, that is a connected subgraph of G ; and, with each edge (u,v) of K we associate a *super-edge*, a path connecting the super-nodes of u to of v. Super-nodes are pairwise disjoint and super-edges can only meet at their super-node endpoints. For $K = K_{2,r}$ we denote by **A** and **B** the super-nodes of the two degree-r vertices of $K_{2,r}$.

2.2. Proof of Lemma 1 : Planar Case

An *LMR-embedding* of a planar graph G is a plane embedding of G with three distinguished paths, namely L, M, R (for left, middle, and right), sharing only their extremities and such that :

1) $L \cup R$ is the border of the outerface of G ;
2) $\textsc{In}(L \cup M)$ and $\textsc{In}(M \cup R)$ contain no vertices ;
3) $\textsc{In}(M \cup R)$ contains no edges with both endpoints in M ;
4) M has length at least two.

Properties 1 and 2 of LMR-embedding imply that the paths L, M, R span the vertices of the graph. See Fig. 1(a) for an example.

Lemma 1 is proved thanks to the following two lemmas.

Lemma 3 *Given an LMR-embedding of G, we can in linear time either extract a $K_{2,4}$-minor, or find a vertex whose removal leaves G outerplane.*

Proof. Let u,v be the common vertices of L, M, R. Observe that, from the definition of LMR-embedding, $G \setminus R]u,v[$ is outerplane. In order to prove the lemma, we apply the following rules whenever it is possible.

2.2.0.1. Rule 1 :

If $\textsc{In}(M \cup R)$ has no edge connecting a vertex of M and of R. Then, removing u makes the new outerface bordering all the vertices of M. Since all the vertices of $L \cup R$ are on the border of the outerface, the embedding $G \setminus \{u\}$ is outerplane.

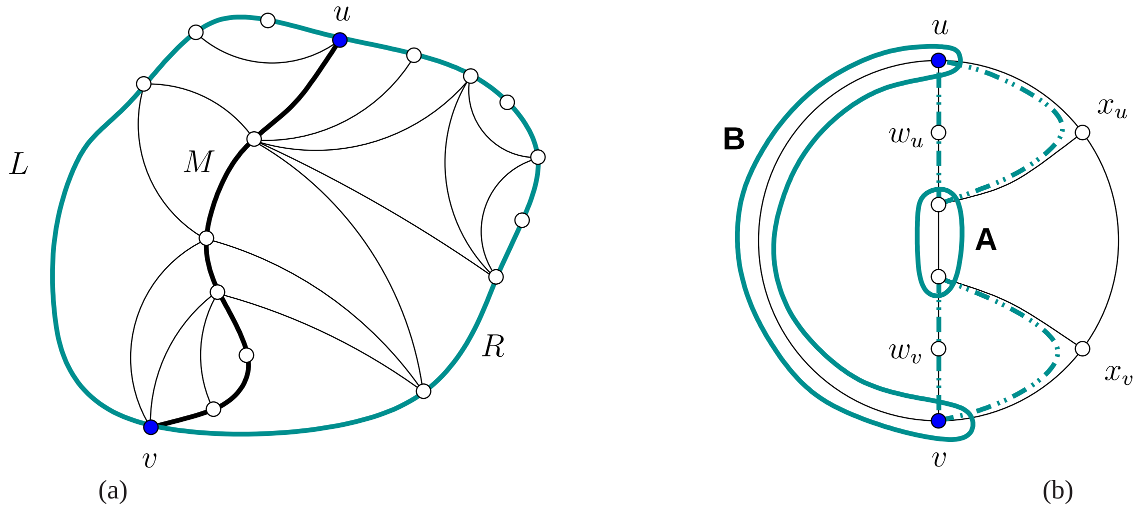


Figure 1 – An LMR-embedding (a). A $K_{2,4}$-minor for the proof of Lemma 3(b).

2.2.0.2. Rule 2 :

If there is a vertex w of R such that all the edges of $\text{IN}(M \cup R)$ incident to a vertex of M are incident to w. Then, removing w makes $G \setminus \{w\}$ outerplane.

2.2.0.3. Rule 3 :

If there is a vertex w adjacent to u in M that is also adjacent to a vertex x of R. Then, we construct a new LMR-embedding and update L, M, R as follows :

– L becomes $L \cup R[u, x]$;
– M becomes $M[v, w] \cup \{wx\}$; and
– R becomes $R[x, v]$.

We apply also this construction, by exchanging the role of u and v, if there is a vertex w adjacent to v in M that is also adjacent to a vertex x of R. Note that in both cases, $|R|$ decreases by at least one vertex.

If none of the rules 1, 2, or 3 applies, then G contains a $K_{2,4}$ minor. Indeed, Rule 3 does not apply, so vertex u has a neighbor w_u in M without any neighbor in R. The same holds for vertex v that has a neighbor w_v in M without any neighbor in R. Because of Rule 1, $M]w_u, w_v[$ contains at least a one vertex which has a neighbor in R. By Rule 2, all the edges from M (and so from $M]w_u, w_v[$) to R are not incident to the same vertex of R. Thus, there are two different vertices x_u and x_v in R with a neighbor in $M]w_u, w_v[$.

Therefore, we can construct a $K_{2,4}$-minor, by defining the super-nodes $\mathbf{A} = M]w_u, w_v[$ and $\mathbf{B} = L[u, v]$, as shown on Fig. 1(b)).

If we are not in the previous case where we have constructed a $K_{2,4}$-minor, then we must end by Rule 1 or Rule 2, and so find a vertex whose removal makes the embedding outerplane.

To conclude, we observe that this can be done overall in linear time, by first applying while possible Rule 3. This can be done by visiting a constant time each edge of the graph. Then, applying Rule 1 and Rule 2 takes a linear time. □

Lemma 4 *Every* 2*-connected planar graph that is not a cycle has an LMR-embedding, or contains a* $K_{2,4}$*-minor. Moreover, such an embedding or such a minor can be constructed in linear time.*

Due to space contraint, the long and technical proof of Lemma 4 has been removed.

Proof of Lemma 1. Clearly, if G is a cycle (that one can check in linear time), then the removal of any vertex leaves G outerplanar. If G is planar but not a cycle, we can apply Lemma 4. In linear time, we find either an LMR-embedding for G, or extract a $K_{2,4}$-minor. If we have obtained an LMR-embedding for G, we can apply Lemma 3 and, in linear time, either find a vertex whose removal leaves G outerplanar, or extract a $K_{2,4}$-minor. This completes the proof of the lemma. □

2.3. Proof of Lemma 2 : Non-Planar Case

In order to proof Lemma 2, we need of the following key lemma.

Lemma 5 *Let H be a subdivision of $K_{3,3}$ that is a subgraph of a 2-connected graph G. The attachments of any connected component of $G \setminus H$ induced an edge of H, or one can find in linear time a $K_{2,4}$-minor in G.*

Proof. The subgraph H is composed of two sets of degree-3 vertices denoted by I and J, and of nine paths denoted by $P[i, j]$ linking any vertex $i \in I$ to any vertex $j \in J$.

Consider a connected component X of $G \setminus H$, and let A be the set of its attachments. We first observe that $|A| > 1$ otherwise the singleton A would be a cut-vertex of G : impossible G is 2-connected.

So consider two attachments $u, v \in A$. The remaining of the proof consists to show that if u and v are at distance at least two in H, then one can find in linear time a $K_{2,4}$-minor in G. Say in other words, if G is $K_{2,4}$-minor free, then the attachments of X form exactly one edge of H.

Let assume that $u \in P[i_u, j_u]$ and $v \in P[i_v, j_v]$ for some $i_u, i_v \in I$ and $j_u, j_v \in J$, and that u, v are at distance at least two in H. Among the four lengths $|P[i_u, u]|$, $|P[u, j_u]|$, $|P[i_v, v]|$, and $|P[v, j_v]|$, we will assume that $|P[i_u, u]|$ is the smallest one. This can be done by possibly exchanging first u and v if the minimum is attained by v, and then by possibly exchanging i's and j's if $|P[i_u, u]| > |P[u, j_u]|$. Two cases occur :

2.3.0.4. Case 1 :

$i_u = i_v$ and $j_u = j_v$.

Both vertices u, v belong to the path $P = P[i_u, j_v]$. Note also that going from i_u to j_v on P we must encounter u before v by minimality of $|P|$. We set $\mathbf{A} = P[i_u, u]$ and $\mathbf{B} = P[v, j_v]$ for the two nodes of the $K_{2,4}$-minor. Between i_u and j_v in there are in H two disjoint paths of length at least two that uses none of the edges of P. The subpath $P[u, v]$ is of length at least two since u, v are at distance at least two in H. And finally, there is a fourth path of length at least two through component X, completing the construction of the $K_{2,4}$-minor.

2.3.0.5. Case 2 :

$i_u \neq i_v$ or $j_u \neq j_v$.

We present the proof only for $i_u \neq i_v$, the proof is similar if $j_u \neq j_v$.

If $v = j_v$, then $u = i_u$ by minimality of $|P[i_u, u]|$. In particular, both vertices u, v belong to the path $P[i_u, j_v]$, and we can conclude by Case 1. So we can assume that the subpath $P[v, j_v]$ contains at least one edge of H. Note also that the subpath $P[u, j_u]$ contains at least one edge as well by minimality of $|P[i_u, u]|$.

We set $\mathbf{A} = P[i_u, u]$ and $\mathbf{B} = P[i_v, v]$ for the degree-4 super-nodes of the $K_{2,4}$-minor. From the above discussion, the subpaths $P[u, j_u]$ and $P[v, j_v]$ contains at one edge, and therefore contracting $\mathbf{A}$ and $\mathbf{B}$ in H still result in a subdivision of $K_{3,3}$, say H'. Between $\mathbf{A}$ and $\mathbf{B}$ in H' we have three disjoint paths of length at least two, so in G. These paths are disjoint of the fourth path through component X, which is of length at least two too, completing the construction of the $K_{2,4}$-minor.

Clearly, from the above case analysis, the $K_{2,4}$-minor can be extracted in linear time. □

Proof of Lemma 2. We need to show that either G, which is non-planar and 2-connected, contains a $K_{2,4}$-minor, or contains a vertex v_0 such that $G \setminus \{v_0\}$ is planar. For that we will analyze a simple procedure called FIND(G) that either returns a $K_{2,4}$-minor or such a vertex v_0. An $O(n)$ time implementation of this procedure is given after the proof of its correctness.

2.3.0.6. Procedure FIND(G) :

1. If $n \leqslant 5$, then return as v_0 any vertex of G.
2. Construct a K_5 or $K_{3,3}$ subdivision H of G.
3. If H is a subdivision of K_5, then return a $K_{2,4}$-minor constructed from H.
4. While $V(H) \neq V(G)$:

4a. Choose any connected component X of $G \setminus H$.

4b. Apply Lemma 5 to G and H. If a $K_{2,4}$-minor is founded, return it. Otherwise replace in H the edge $\{u, v\}$ induced by the attachments of X by a path from u to v through X.

5. Let v_0 be any degree-3 vertex of H. If $G \setminus \{v_0\}$ is planar, return v_0.
6. Construct a K_5 or $K_{3,3}$ subdivision H' of $G \setminus \{v_0\}$.
7. If H' is a subdivision of K_5, then return a $K_{2,4}$-minor constructed from H'.
8. Return the $K_{2,4}$-minor constructed whenever applying Lemma 5 to G and H'.

2.3.0.7. Correctness.

If we end at Step 1, then $G \setminus \{v_0\}$ has at most 4 vertices, and thus is planar. So Step 1 is correct.

As G is not planar, from the Kuratowski's criterion [30, 31], G must contain a subgraph *homeomorphic* to either K_5 or $K_{3,3}$, that is a subdivision of K_5 or of $K_{3,3}$. Such a subgraph H at Step 2 can be constructed in $O(n)$ time [32, 33]. If H is a subdivision of K_5 (Step 3), we first extend it to a proper subdivision $\tilde{H}$ of K_5, i.e., a subdivision $\tilde{H} \neq K_5$. Assume $H = K_5$, otherwise we directly set $\tilde{H} = H$. Note that $G \neq H$ since $n > 5$, so $G \setminus H$ contains at least one vertex. Since G is 2-connected, there must exists two vertices of H connected by a path in $G \setminus E(H)$. This path can be constructed by testing connectivity between the 10 pairs of vertices of H. This path and H form a proper subdivision $\tilde{H}$ of K_5. Now we construct a $K_{2,4}$-minor in $\tilde{H}$, i.e., a model, as follows. Let X be the set of the five degree-4 vertices in H (corresponding to vertices of K_5). Since $\tilde{H}$ is a proper subdivision of K_5, there exists $a, b \in X$ connected in $\tilde{H}$ by a path P of length at least two. We set $\mathbf{A} = \{a\}$ and $\mathbf{B} = \{b\}$. In $\tilde{H}$, between a and b, there are 3 paths, each one through a distinct vertex of $X \setminus \{a, b\}$. These paths are of length at least two, pairwise disjoint, and disjoint from P too. Path P is the fourth one. This completes the construction of the $K_{2,4}$-minor, and proves the correctness of Step 3. Clearly, the above construction can be done in time linear in the size of $\tilde{H}$, which is $O(n)$.

We are left with the case where H is a subdivision of $K_{3,3}$ in Step 4. We observe that, if we do not end with a $K_{2,4}$-minor at Step 4b, then the number of vertices of H increases by at least one (one edge is replaced by a path of length at least two through X). It turns out that either we end at Step 4b and return a $K_{2,4}$-minor, or we are left at Step 5 with a spanning subgraph H which is a subdivision of $K_{3,3}$.

Let $G' = G \setminus \{v_0\}$, where v_0 is a degree-3 vertex of H. Assume that G' is non-planar, i.e., we did not end at Step 5. Step 6 and 7 are correct as they are the same as Step 2 and 3 by replacing H by H' (note that $H' \neq G$). At Step 8 we can apply Lemma 5 to G and H' because G is 2-connected. We claim that the application of this lemma returns a $K_{2,4}$-minor of G.

Consider the connected component X_0 of $G \setminus H'$ containing v_0. If we do not return a $K_{2,4}$-minor, then the attachments u, v of X_0 induced an edge in H'. In particular $\{u, v\}$ disconnects G into $X_0 \ni v_0$ and another component, say Y, containing the other vertices of H' (there are at least four such vertices). It follows that $\{u, v\}$ is a v_0-separator in G. However, in H, there is a path from v_0 to Y. So $\{u, v\}$ is not a v_0-separator in H and thus not in G as well : a contradiction. So Step 8 is correct, completing the proof of the correctness of Procedure $\text{FIND}(G)$.

2.3.0.8. An $O(n)$ time algorithm.

Procedure $\text{FIND}(G)$ cannot be directly used in order to get an $O(n)$ time complexity algorithm, and this for at least two reasons. First, the non-planar input graph G may have $m = \Omega(n^2)$ edges. Secondly, the while-loop (Step 4) may require $\Theta(n)$ loops, each one requiring $\Theta(m)$ time.

Using the bound of [27], we know that if G has more than $(n-1)(r+1)/2 = 2.5n - O(1)$ edges, then G must contain a $K_{2,4}$ minor. However, it does not imply that Procedure $\text{FIND}(G)$ returns such a minor. This is due to the fact that this procedure could instead find a vertex v (of high degree) so that $G \setminus \{v\}$ is planar, despite G has $> 2.5n$ edges.

However, if G has at least $4n - 6$ edges, then the application of $\text{FIND}(G)$ necessarily returns a $K_{2,4}$-minor. This is because $G \setminus \{v\}$, for any vertex v, cannot be planar : $G \setminus \{v\}$ contains at least $4n-6-(n-1) = 3n-5$ edges. In other words, if G has too many edges, we can concentrate our attention to any subgraph of G with $4n-6$ edges, and apply on it a fast implementation of FIND. The subgraph extraction can be done in $O(n)$ time by selecting its $4n-6$ first edges. So we can safely assume that G has $O(n)$ edges.

In the proof of the correctness of FIND, we have seen that each steps, but Step 4, takes a linear time, so $O(n)$ time. In Step 4b, it takes $O(n)$ if a $K_{2,4}$-minor is constructed. Otherwise, by the 2-connectivity of G, a simple DFS from vertices of $G[X \cup \{u, v\}]$ will find out a path from u to v in time proportional to the length of the path. In other words, each edge is visited $O(1)$ time, and so the while-loop has total cost $O(n)$. This completes the proof of Lemma 2. □

3. On Generalizing Theorem 1

The Jørgensen's conjecture states that every 6-connected K_6-minor free graph has a vertex whose removal leaves the graph planar. The conjecture implies the Hadwiger's conjecture for $r = 6$, about $(r-1)$-colorability of K_r-minor free graphs. Actually, Robin Thomas proposed the following generalization :

Conjecture 2 (R. Thomas) *For each $r \geqslant 5$, there is a constant $g(r)$ such that if G is r-connected with at least $g(r)$ vertices, then either G has a K_r-minor or G has a set X of $r-5$ vertices such that $G \setminus X$ is planar.*

Note that the condition on $g(r)$ is required as there are graphs with $\Omega(r\sqrt{\log r} \cdot n)$ edges [34, **?**] that are K_r-minor free, and so that they cannot have $O(n)$ edges like any planar graph by the removal of $O(r)$ vertices. So $g(r) = \Omega(r\sqrt{\log r})$.

In the light of our result, we propose the following conjecture :

Conjecture 3 *For each $r \geqslant 2$, there is a constant $f(r)$ such that if G is $f(r)$-connected, then either G has a $K_{2,r}$-minor or G has a set X of $r-2$ vertices such that $G \setminus X$ is outerplanar.*

The condition on the minimum number of vertices is not required in this latter conjecture since $K_{2,r}$-minor free graphs have no more than $O(rn)$ edges [27]. We show (see [35] for a proof) :

Proposition 1 *Conjecture 3 is true for $r = 2, 3, 4$ and with $f(2) = 0$, and $f(3) = f(4) = 2$.*

We have proved the first values of r for Conjecture 3 with $f(3) = f(4) = 2$. Note that if $f(r) < 2$, then, for any value of $r \geqslant 3$, the conjecture becomes wrong by considering, for instance, a chain of K_{r+1}. So, the value $f(r)$ given in Proposition 1 is the lowest possible one for each $r \in \{2, 3, 4\}$.

We are unable to prove the conjecture for $r = 5$. However, if it is true, we must have $f(5) \geqslant 3$ as shown by the next result (see [35] for its proof).

Theorem 3 *For every integer $k \geqslant 0$, there is a 2-connected $K_{2,5}$-minor free graph that requires the removal of at least k vertices to leave the graph outerplanar. Moreover this graph is planar and has $4k + 4$ vertices.*

4. Treewidth of Minor Free Planar Graphs

Bounding the treewidth of a graph by a function of a minor it excludes is one of the most surprising property of the Graph Minor Theory. In a seminal paper, Robertson et al. [18], have showed that the treewidth of a graph is bounded if the graph excludes a finite planar minor.

We investigate this question for planar graphs. It is known [18] that the treewidth of a planar graph excluding a r-vertex planar as a minor is $O(r)$. This result is prove by combining the following two results :

Lemma 6 ([18])

(i) If G is planar and excludes an $r \times r$-grid minor, then its treewidth is at most $6r - 5$.
(ii) If H is a r-vertex planar graph, then it is a minor of a $(14r - 24) \times (14r - 24)$-grid.

So, if a planar graph G excludes a planar graph H as minor, then by (ii) G excludes a $14r \times 14r$-grid, and by (i) the treewidth of G is $< 84r$. This bound cannot be asymptotically improved in general as there are r-vertex planar graphs H that are not minor of the $r \times r$ grid which has treewidth r. For instance, consider $H = \mathcal{C}_r$ the pathwidth-3 graph obtained from Cartesian product of a K_3 and a r-vertex path (it can be drawn as r nested triangles). However, there is hope to improved this linear bound if we restrict the family of excluding minors.

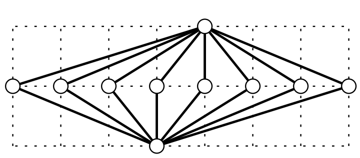
More formally, we are looking for suitable graph families $\mathcal{F}$ and functions t such that the treewidth of every planar graph excluding an r-vertex graph minor of $\mathcal{F}$ is at most $t(r)$. Note that $\mathcal{C}_r \in \mathcal{F}$ forces $t(r) \geqslant r$.

We will determine a large family $\mathcal{F}$ for which $t(r) = O(\sqrt{r}\,)$, namely the family of all graphs having a poly-line $O(1) \times r$-grid drawing.

A graph H has a poly-line $p \times q$-grid drawing if H has a plane drawing such that vertices are plot at the vertices of the $p \times q$ grid, and edges are contiguous sequences of segments, each segment being a straight-line between two vertices of the $p \times q$ grid (see [36] for a wide overview). The drawing is *straight-line* if each edge consists of one segment only.

Due to space contraint, the proof of the next result is removed (see [35]).

Theorem 4 *The treewidth of every planar graph excluding a graph having a poly-line $p \times q$-grid drawing is $O(p^{3/2}\sqrt{q}\,)$.*



As depict on the above figure, $K_{2,r}$ has a straight-line $3 \times r$-grid drawing. Hence,

Corollary 1 *The treewidth of every planar $K_{2,r}$-minor free graphs is $O(\sqrt{r}\,)$.*

It is maybe worth to mention that the family of graphs having a straight-line $p \times q$-grid drawing is not closed under minor taking, for each $p \geqslant 3$ (see [37]), whereas the family of pathwidth-p graphs is. In [37], it is also proved that the pathwidth is a lower bound on the number of rows in any grid drawing of a tree. Connections between straight-line 3D-grid drawing and Minor Graph Theory are given [38].

The useful Theorem 4 allow us the plug results from literature of Graph Drawing Theory. Theorem 4 applies in particular to r-vertex trees. They have straight-line $(4+\log_2 r)\times r$-grid drawing (cf. [36, 39]), and more generally, trees of pathwidth k have straight-line $(2k-1)\times r$-grid drawings [40]. Recently, Biedl [41] has showed that r-vertex outerplanar graph has poly-line $O(\log r) \times O(r)$-grid drawing. Therefore,

Corollary 2 *The treewidth of every planar H-minor free graph has treewidth :*

- *$O(k^{3/2}\sqrt{r}\,)$ if H is an r-vertex tree of pathwidth k ; and*
- *$O(\sqrt{r}\log^{3/2} r\,)$ if H is an r-vertex outerplanar graph.*

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